Comparing Hierarchical Network Partitions Based on Relative Entropy

LoG Local Meetup Aachen 2024

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Comparing Partitions

Comparing Partitions

Jaccard index $J(a, b) = \frac{|a \cap b|}{|a \cup b|}$ $\quad J(A, B) = \sum_{a \in A} \frac{|a|}{n} \max_{b \in B} J(a, b)$

Reference: A

B $\quad J(A, B) = \frac{1}{2}$

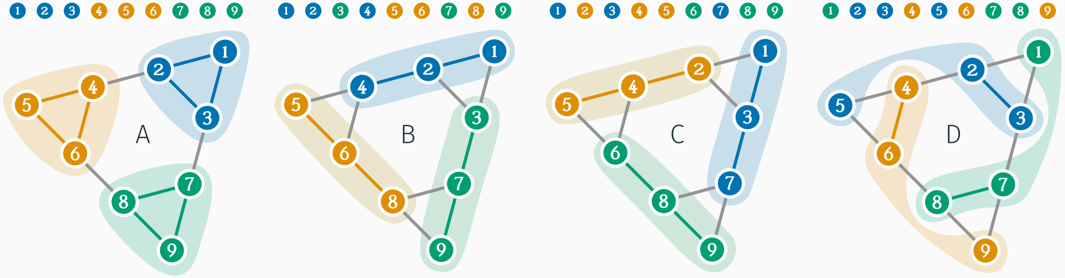
C $\quad J(A, C) = \frac{1}{2}$

D $\quad J(A, D) = \frac{1}{2}$

Similar for mutual information etc.

Comparing *Network* Partitions

Comparing Network Partitions



$$J(\mathrm{A},\mathrm{B}) = J(\mathrm{A},\mathrm{C}) = J(\mathrm{A},\mathrm{D}) = \frac{1}{2}$$

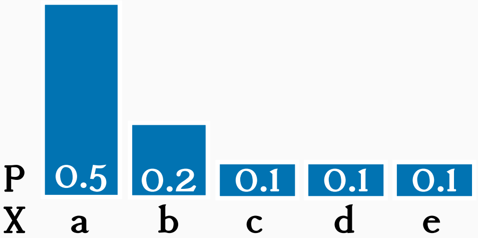
But what about the links?

Idea:
KL divergence + random walks

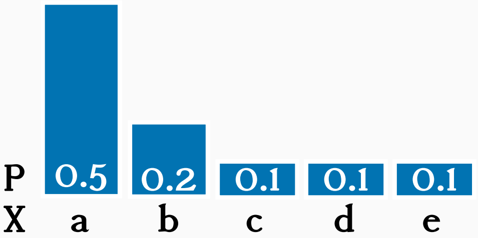
Kullback-Leibler divergence

X **a** **b** **c** **d** **e**

Kullback-Leibler divergence

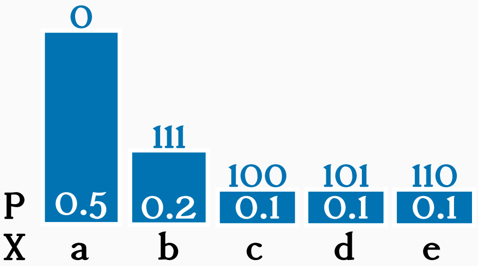


Kullback-Leibler divergence



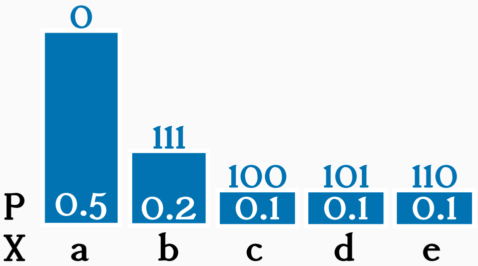
a a c d b b a b e c

Kullback-Leibler divergence



a a c d b b a b e c

Kullback-Leibler divergence



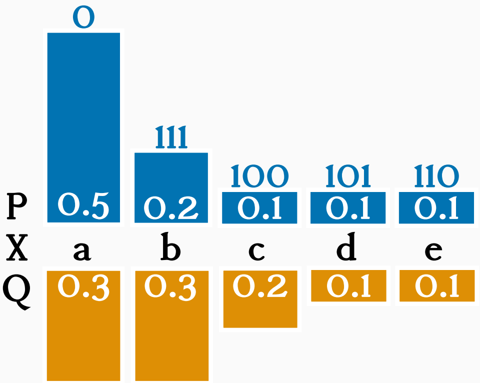
$$H(P) = -\sum_{x \in X} p_x \log_2 p_x$$

$$H(P) \approx 1.96 \text{ bits}$$

0 0 100 101 111 111 0 111 110 100

a a c d b b a b e c

Kullback-Leibler divergence



0 0 100 101 111 111 0 111 110 100

a a c d b b a b e c

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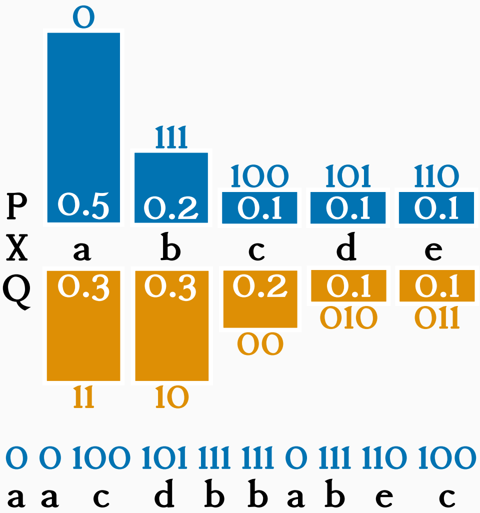
Kullback-Leibler divergence



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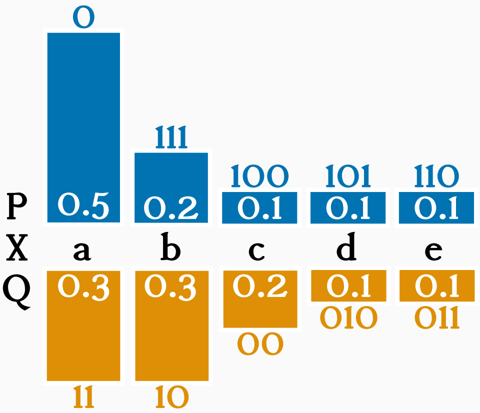
Kullback-Leibler divergence



0 0 100 101 111 111 0 111 110 100

a a c d b b a b e c

11 11 00 010 10 10 11 10 011 00

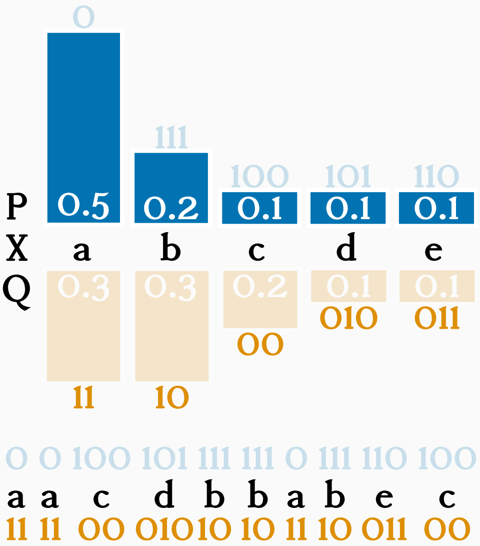
$$H(P) = -\sum_{x \in X} p_x \log_2 p_x$$

$$H(P) \approx 1.96 \text{ bits}$$

$$H(Q) \approx 2.17 \text{ bits}$$

$$D_{KL}(P \,||\, Q) = \sum_{x \in X} p_x \log_2 \frac{p_x}{q_x}$$

Kullback-Leibler divergence



$$H(P) = -\sum_{x \in X} p_x \log_2 p_x$$

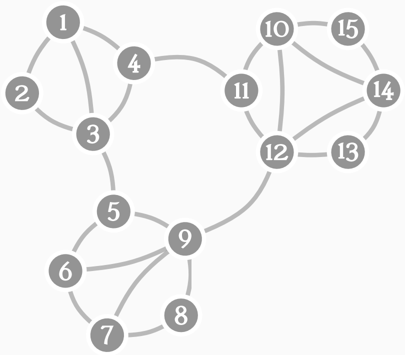
$$H(P) \approx 1.96 \text{ bits}$$

$$H(Q) \approx 2.17 \text{ bits}$$

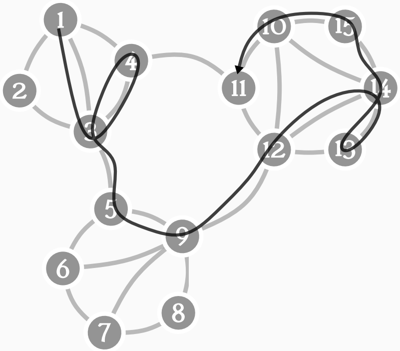
$$D_{KL}(P \,||\, Q) = \sum_{x \in X} p_x \log_2 \frac{p_x}{q_x}$$

$$D_{KL}(P \,||\, Q) \approx 0.15 \text{ bits}$$

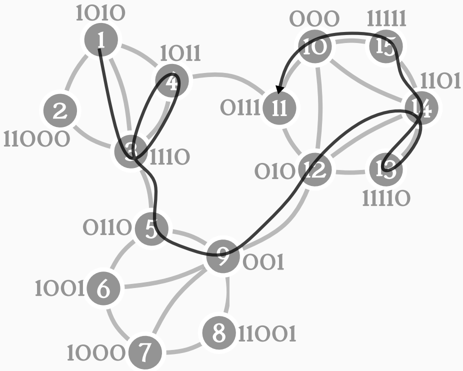
Random Walks

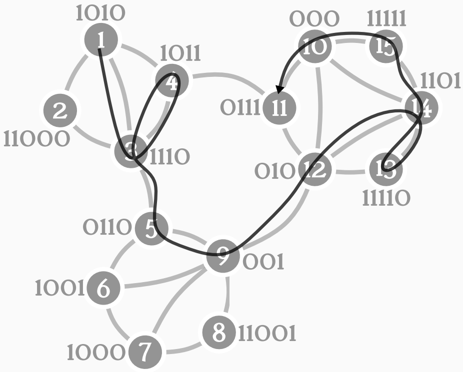


Random Walks



Random Walks

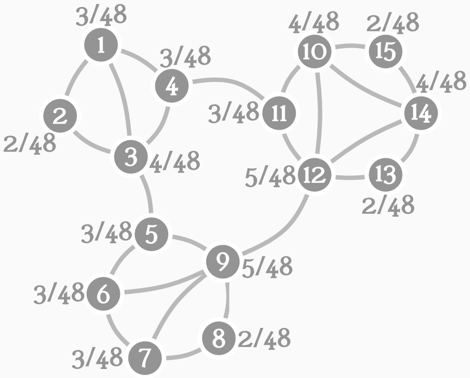




1010 1110 1011 1110 0110 001 010
1101 11110 1101 11111 000 0111

Description length

$$H(P) = -\sum_{v \in V} p_v \log_2 p_v \text{ bits} \tag{1}$$



Description length

$$H(P) = -\sum_{v \in V} p_v \log_2 p_v \text{ bits} \tag{1}$$

Visit rates

$$p_v = \sum_{u \in V} p_u t_{uv} \qquad t_{uv} = \frac{w_{uv}}{\sum_{v \in V} w_{uv}} \tag{2}$$

Combine (1)+(2)

$$H(P) = -\sum_{u \in V} p_u \sum_{v \in V} t_{uv} \log_2 p_v$$

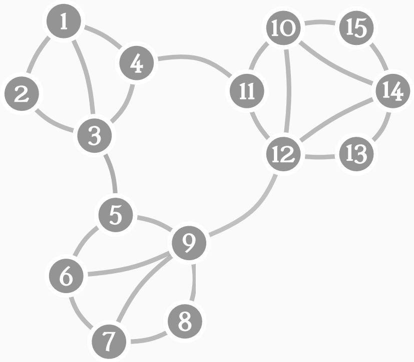
But we are interested in communities...

$$H(M) = -\sum_{u \in V} p_u \sum_{v \in V} t_{uv} \log_2 s(M, u, v)$$

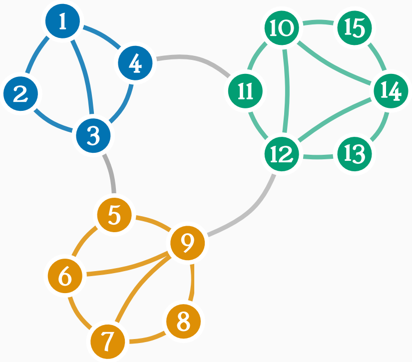
The Map Equation

$\longrightarrow$ Rosvall and Bergstrom; PNAS 2008

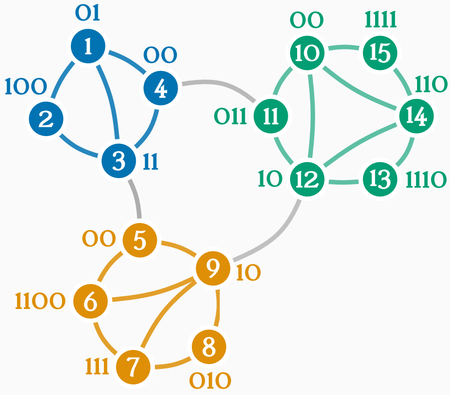
The Map Equation



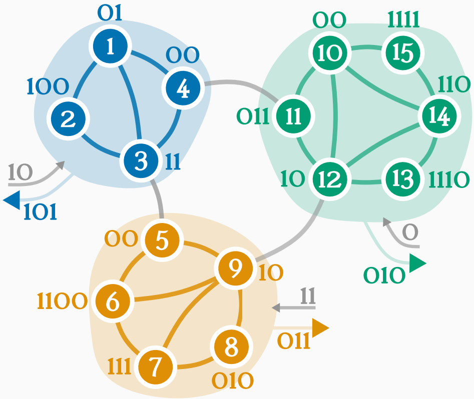
The Map Equation



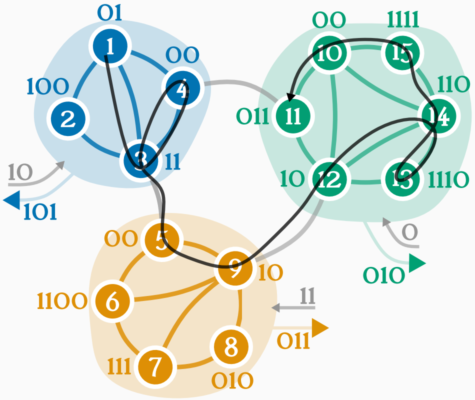
The Map Equation

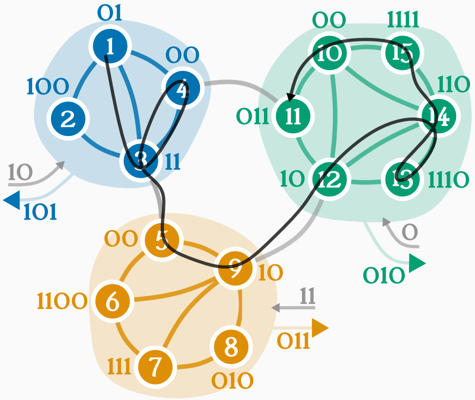


The Map Equation



The Map Equation



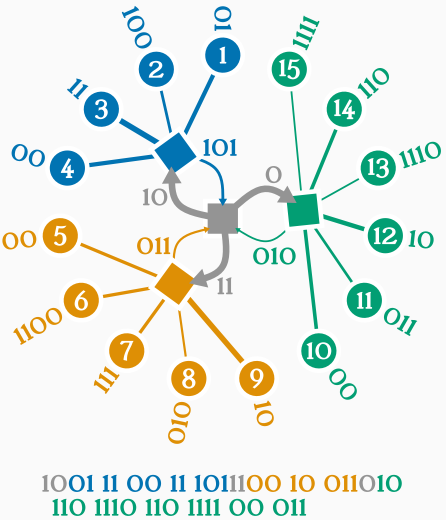


1001 11 00 11 1011100 10 011010
110 1110 110 1111 00 011

$$L(M) = qH(Q) + \sum_{m \in M} p_m H(P_m) \rightarrow \min$$

$$= -\sum_{u \in V} p_u \sum_{v \in V} t_{uv} \log_2 \text{mapsim}(M, u, v)$$

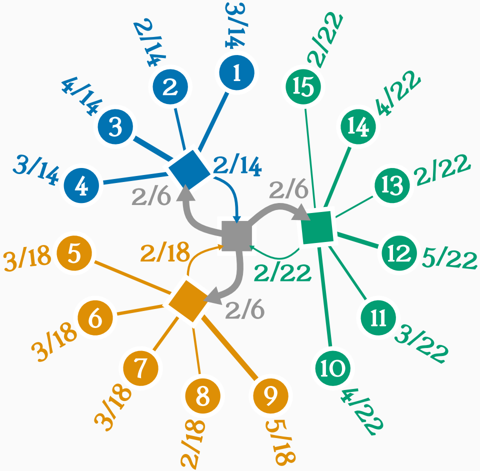
The Map Equation



$$L(\mathsf{M}) = qH(Q) + \sum_{\mathsf{m} \in \mathsf{M}} p_{\mathsf{m}} H(P_{\mathsf{m}})$$

$$= -\sum_{u \in V} p_u \sum_{v \in V} t_{uv} \log_2 \operatorname{mapsim}(\mathsf{M}, u, v)$$

The Map Equation



$$L(\mathsf{M}) = qH(Q) + \sum_{\mathsf{m}\in\mathsf{M}} p_{\mathsf{m}} H(P_{\mathsf{m}})$$

$$= -\sum_{u\in V} p_u \sum_{v\in V} t_{uv} \log_2 \operatorname{mapsim}(\mathsf{M}, u, v)$$

$$\operatorname{mapsim}(\mathsf{M}, u, v) = \begin{cases} \dfrac{p_v}{p_{\mathsf{m}_v}} & \text{if } \mathsf{m}_u = \mathsf{m}_v, \\ \dfrac{\mathsf{m}_{u,\text{exit}}}{p_{\mathsf{m}_u}} \cdot \dfrac{q_{\mathsf{m}_v}}{q} \cdot \dfrac{p_v}{p_{\mathsf{m}_v}} & \text{if } \mathsf{m}_u \neq \mathsf{m}_v, \end{cases}$$

→ Blöcker et al.; PMLR 2022

Flow Divergence

Flow Divergence

Idea

Measure how many extra bits we need to describe a random walk using an “estimate” B of the networks “true” partition A.

$$D_{KL}(P \,||\, Q) = \sum_{x \in X} p_x \log_2 \frac{p_x}{q_x} \qquad L(\mathrm{M}) = -\sum_{u \in V} p_u \sum_{v \in V} t_{uv} \log_2 \mathbf{mapsim}(\mathrm{M}, u, v)$$

$$D_F(\mathrm{A} \,||\, \mathrm{B}) = \sum_{u \in V} p_u \sum_{v \in V} t_{uv}^{\mathrm{A}} \log_2 \frac{\mathbf{mapsim}(\mathrm{A}, u, v)}{\mathbf{mapsim}(\mathrm{B}, u, v)}$$

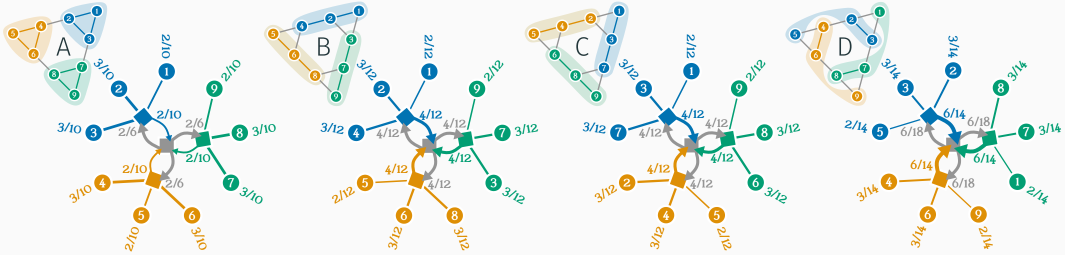
$$t_{uv}^{\mathrm{A}} = \frac{\mathbf{mapsim}(\mathrm{A}, u, v)}{\sum_{v(\neq u)} \mathbf{mapsim}(\mathrm{A}, u, v)}$$

Complexity $\mathcal{O}(m \cdot n)$ using the regularities in the partitions

m : number of communities $\qquad$ n : number of nodes

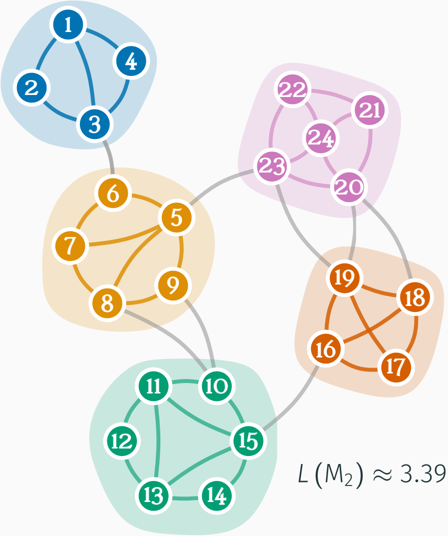
Flow Divergence

$L(\mathrm{A}) \approx 2.86$ bits $\quad L(\mathrm{B}) \approx 3.73$ bits $\quad L(\mathrm{C}) \approx 3.73$ bits $\quad L(\mathrm{D}) \approx 4.47$ bits

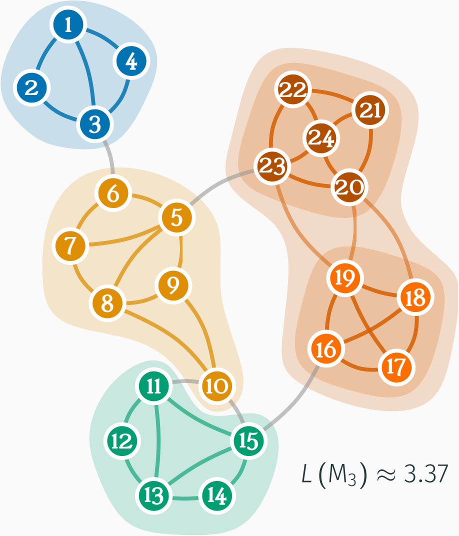


	Other			
Reference	A	B	C	D
A	0	1.92	1.92	1.5
B	1.8	0	1.17	1.99
C	1.8	1.17	0	1.48
D	1.14	1.78	1.25	0

Flow Divergence

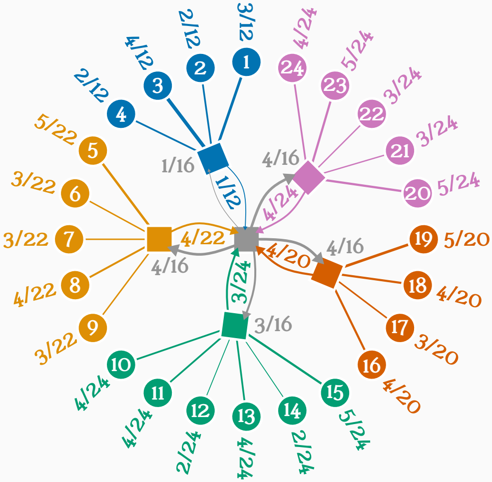


$L(M_2) \approx 3.39$ bits

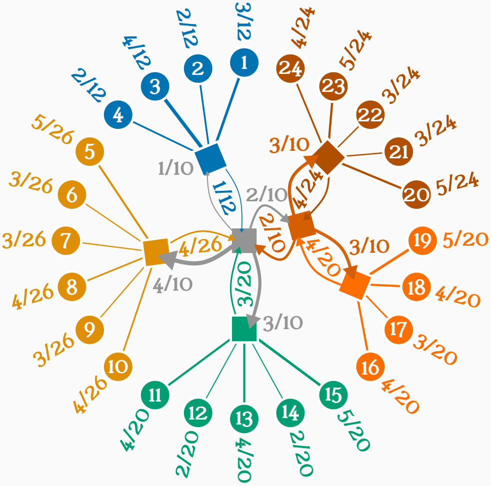


$L(M_3) \approx 3.37$ bits

Flow Divergence



$D_F\,(M_2\,||\,M_3) \approx 0.11$ bits



$D_F\,(M_3\,||\,M_2) \approx 0.01$ bits

Conclusion

Conclusion

Motivation

Communities are based on link patterns
but common partition-similarity scores ignore links.

Our Solution

We combine the ideas behind describing random walks and the KL-divergence to develop a link-aware partition-similarity score.

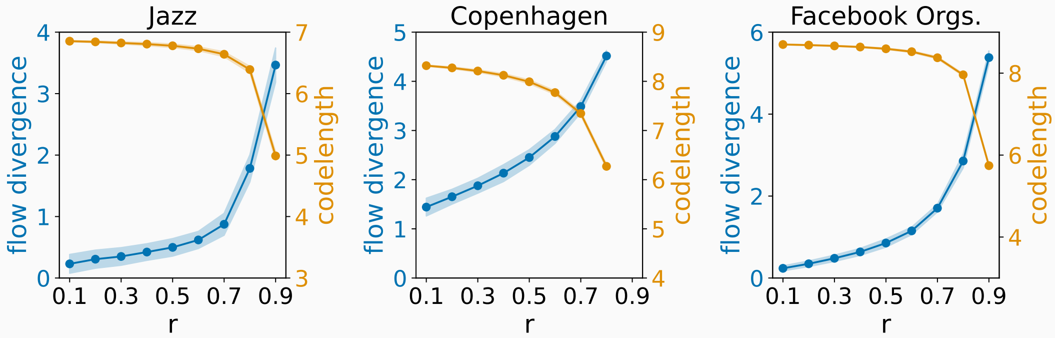
- asymmetric
- does not require matching of communities between partitions
- naturally compares hierarchical partitions

Thank you for your attention!

Check out our preprint: arXiv:2401.09052

Appendix

Flow Divergence



Flow Divergence

(a) Flow Divergence (b) Jaccard Index (c) Adjusted Mutual Information